

## LETTER TO THE EDITOR

**Heisenberg picture approach to the invariants and the exact quantum motions for coupled parametric oscillators**Jeong-Young Ji<sup>†</sup> and Jongbae Hong<sup>‡</sup>

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**Abstract.** For  $N$ -coupled generalized time-dependent oscillators, primary invariants and a generalized invariant are found in terms of classical solutions. Exact quantum motions satisfying the Heisenberg equation of motion are also found. For number states and coherent states of the generalized invariant, the uncertainties in positions and momenta are obtained.

In the time-dependent coupled oscillator system, the invariant method is powerful in analysing the quantum mechanical behaviours. The Lewis–Riesenfeld (LR) invariant [1, 2] has been derived by various methods such as time-dependent canonical transformations [3–5], the Noether theorem [6], and Ermakov’s technique [7]. While the LR invariant is quadratic in position and momentum operators, the primary invariant found in [8] is linear in the operators.

The primary invariant is so simple in structure that it may be useful in studying time-dependent coupled oscillators. Recently, the Ermakov–Lewis invariant has been constructed using amplitude–phase decomposition in a coordinate–coordinate coupled system [9]. For the most general form of coupled oscillators which includes any couplings of coordinates and momenta, the LR-type invariant was found using the canonical transformation [3] and the first-order invariant was constructed using the Noether theorem [10].

In the Heisenberg picture, the quantum motions of position and momentum in a single oscillator system have been found in [11], where the LR invariant exhibits the time-independency explicitly. In this letter, we extend the previous work [11] to the coupled oscillators. Since the coupled parametric oscillator system studied here is the most general form, our results will be applied to the studies in quantum optics as well as in atomic and molecular physics. We construct primary invariant and LR invariant in terms of classical solutions and find the time-evolutions of position and momentum operators which are the solutions of the Heisenberg equations.

Let us consider a general oscillator-type Hamiltonian

$$H(t) = A_{\mu\nu}(t)z^\mu z^\nu + B_\mu(t)z^\mu + C(t) \quad (1)$$

where the matrix  $A_{\mu\nu}$  is real and symmetric. As a unified notation for the coordinates in phase space, we use  $\{z^\mu\}$ :  $q_i = z^i$ ,  $p_i = z^{N+i}$ . Here all Greek indices  $\alpha, \beta, \dots$  range from

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1 to  $2N$  and Latin indices  $i, j, \dots$  range from 1 to  $N$ . The symplectic matrix  $[\epsilon^{\mu\nu}]$  and its inverse matrix  $[\epsilon_{\mu\nu}]$  are defined by

$$\epsilon \equiv [\epsilon^{\mu\nu}] = \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix} \quad [\epsilon_{\mu\nu}] = \begin{pmatrix} 0 & -1_N \\ 1_N & 0 \end{pmatrix} \quad (2)$$

where  $1_N$  is an  $N \times N$  identity matrix.

We look for the primary invariant of the form

$$b = v_\nu(t)z^\nu(t) + u(t) \quad (3)$$

which satisfies the invariant equation

$$\frac{\partial}{\partial t} b(t) - i[b(t), H(t)] = 0. \quad (4)$$

From this invariant equation and the commutation relation  $[z^\mu, z^\nu] = i\epsilon^{\mu\nu}$ , we have a system of the first-order differential equations for  $v_\nu$  and  $u$ :

$$\dot{v}_\nu + 2v_\sigma \epsilon^{\sigma\rho} A_{\rho\nu} = 0 \quad (5)$$

$$\dot{u} + v_\nu \epsilon^{\nu\sigma} B_\sigma = 0. \quad (6)$$

Note that equation (5) is identical to the homogeneous part of the classical equation of motion (Hamilton's equation) for equation (1):

$$\dot{z}_{\text{cl},\nu} = -2z_{\text{cl},\sigma} \epsilon^{\sigma\rho} A_{\rho\nu} + B_\nu \quad (7)$$

with  $z_{\text{cl},\nu} = \epsilon_{\nu\rho} z_{\text{cl}}^\rho$ . For equation (6), the solution of  $u$  is easily found by the direct integration

$$u(t) = u(0) - \int_0^t v_\nu(s) \epsilon^{\nu\sigma} B_\sigma(s) ds. \quad (8)$$

If we represent the solution of equation (5) as a complex row vector, there exist  $2N$  linearly independent solutions which we label  $v^{(1)}(t), \dots, v^{(2n)}(t)$ . Combining these solutions together we define the solution matrix

$$V = [v_\nu^\mu] = \begin{pmatrix} v^{(1)} \\ \vdots \\ v^{(2n)} \end{pmatrix} \quad (9)$$

which obeys

$$\dot{V} = -2V\epsilon A. \quad (10)$$

This solution matrix is not determined uniquely because the linear combination of solutions is also a solution. In other words, if  $V$  is a solution matrix, so is  $CV$  with a nonsingular constant matrix  $C$ . We choose the solution matrix of the form:

$$V = i \begin{pmatrix} -\pi^* & \phi^* \\ \pi & -\phi \end{pmatrix} \quad (11)$$

satisfying the following initial conditions with arbitrary parameters  $\omega_i$ ,

$$\phi_{ij}(0) = \frac{1}{\sqrt{2\omega_i}} \delta_{ij} \quad \pi_{ij}(0) = -i\sqrt{\frac{\omega_i}{2}} \delta_{ij} \quad (12)$$

where  $\phi = [\phi_{ij}]$  and  $\pi = [\pi_{ij}]$  are  $N \times N$  matrices. Then we have the following  $2N$ -primary invariants

$$b^\mu = v_\nu^\mu(t)z^\nu(t) + u^\mu(t) \quad (13)$$

which satisfy (i)  $b^i = b_i$ ,  $b^{N+i} = b_i^\dagger$  (ii)  $[b^\mu, b^\nu] = v_\alpha^\mu v_\beta^\nu i\epsilon^{\alpha\beta} = \epsilon^{\mu\nu}$ . These conditions (i) and (ii) mean that we can interpret  $b^i$  ( $b^{i\dagger}$ ) as the annihilation (creation) operator. Inversely, position and momentum operators are given by

$$q_i(t) = \phi_{ij}(t)b_j - \phi_{ij}(t)u_j(t) + \text{h.c.} \quad (14)$$

$$p_i(t) = \pi_{ij}(t)b_j - \pi_{ij}(t)u_j(t) + \text{h.c.} \quad (15)$$

where we have used

$$V^{-1} = i\epsilon V^T \epsilon^T = \begin{pmatrix} \phi & \phi^* \\ \pi & \pi^* \end{pmatrix}. \quad (16)$$

With the first-order invariants, the LR-type invariant is constructed as

$$I = \sum_i^N \omega_i \left( b_i^\dagger b_i + \frac{1}{2} \right). \quad (17)$$

The eigenstates of the invariant (17) are the number states

$$|\mathbf{n}\rangle \equiv |n_1, \dots, n_N\rangle = \prod_i \frac{b_i^{\dagger n_i}}{\sqrt{n_i!}} |\mathbf{0}\rangle \quad (18)$$

where the state  $|\mathbf{0}\rangle$  is defined, as usual, by

$$b_i |\mathbf{0}\rangle = 0 \quad \text{for } i = 1, \dots, N. \quad (19)$$

Further, we define the coherent state as

$$|\alpha\rangle = \prod_{i=1}^N e^{-|\alpha_i|^2/2} \sum_{k_i=0}^{\infty} \frac{\alpha_i^{k_i}}{\sqrt{k_i!}} |k_i\rangle \quad (20)$$

which satisfies

$$b_i |\alpha\rangle = \alpha_i |\alpha\rangle. \quad (21)$$

In the Heisenberg picture the time evolution of the system is described by the time evolution of the quantum operators. By equating the invariants (13) at two different times:

$$v_\nu^\mu(t) z^\nu(t) + u^\mu(t) = v_\nu^\mu(0) z^\nu(0) + u^\mu(0) \quad (22)$$

we deduce the quantum evolution of the Heisenberg operators

$$\mathbf{z}(t) = V^{-1}(t)V(0)\mathbf{z}(0) - V^{-1}(t)[\mathbf{u}(t) - \mathbf{u}(0)]. \quad (23)$$

By direct differentiation, it is easily checked that  $z^\mu(t)$  satisfies the Heisenberg equation of motion  $i\frac{d}{dt}z^\mu = [z^\mu, H]$ . In the explicit form of positions and momenta, we obtain

$$q_i(t) = -i\phi_{ij}(t)\pi_{jk}^*(0)q_k(0) + i\phi_{ij}(t)\phi_{jk}^*(0)p_k(0) - \phi_{ij}(t)[u_j(t) - u_j(0)] + \text{h.c.} \quad (24)$$

$$p_i(t) = -i\pi_{ij}(t)\pi_{jk}^*(0)q_k(0) + i\pi_{ij}(t)\phi_{jk}^*(0)p_k(0) - \pi_{ij}(t)[u_j(t) - u_j(0)] + \text{h.c.} \quad (25)$$

These results can be easily obtained from (15) replacing  $b_i$  by (13) at  $t = 0$ .

Now we examine the quantum properties of the eigenstate and the coherent state. The variations in position and momentum are given by, for the number state (18),

$$\langle \mathbf{n} | (\Delta q_i)^2(t) | \mathbf{n} \rangle = \sum_j (2n_j + 1) |\phi_{ij}(t)|^2 \quad (26)$$

$$\langle \mathbf{n} | (\Delta p_i)^2(t) | \mathbf{n} \rangle = \sum_j (2n_j + 1) |\pi_{ij}(t)|^2 \quad (27)$$

and for the coherent state (20)

$$\langle \alpha | (\Delta q_i)^2(t) | \alpha \rangle = \sum_j |\phi_{ij}(t)|^2 \quad (28)$$

$$\langle \alpha | (\Delta p_i)^2(t) | \alpha \rangle = \sum_j |\pi_{ij}(t)|^2. \quad (29)$$

Furthermore the expectation values of the position and the momentum for the coherent state are

$$\langle \alpha | q_i(t) | \alpha \rangle = \phi_{ij}(t) \alpha_j - \phi_{ij}(t) u_j(t) + \text{c.c.} \quad (30)$$

$$\langle \alpha | p_i(t) | \alpha \rangle = \pi_{ij}(t) \alpha_j - \pi_{ij}(t) u_j(t) + \text{c.c.} \quad (31)$$

which are the same as those of the classical motion.

Finally we suggest that our formalism can be applied to the following type of Hamiltonian

$$H(t) = \mathcal{A}_{\mu\nu} a^\mu a^\nu + \mathcal{B}_\mu a^\mu + C \quad (32)$$

where

$$\mathcal{A}_{\mu\nu} = A_{\rho\sigma} \Lambda_\mu^\rho \Lambda_\nu^\sigma, \mathcal{B}_\mu = B_\rho \Lambda_\mu^\rho. \quad (33)$$

Here we have introduced the creation and annihilation operators defined with an arbitrary parameter  $\lambda$  by

$$q_i = \sqrt{\frac{1}{2\lambda}} (a_i + a_i^\dagger) \quad (34)$$

$$p_i = \frac{1}{i} \sqrt{\frac{\lambda}{2}} (a_i - a_i^\dagger) \quad (35)$$

or in the unified notation ( $a_i = a^i, a_i^\dagger = a^{N+i}$ )

$$z^\mu = \Lambda_\nu^\mu a^\nu \quad (36)$$

with

$$\Lambda = \begin{pmatrix} \sqrt{\frac{1}{2\lambda}} 1_N & \sqrt{\frac{1}{2\lambda}} 1_N \\ \frac{1}{i} \sqrt{\frac{\lambda}{2}} 1_N & -\frac{1}{i} \sqrt{\frac{\lambda}{2}} 1_N \end{pmatrix}. \quad (37)$$

In summary, we find the primary (first-) and second-order (LR-type) invariants of the  $N$ -coupled generalized time-dependent oscillators. Using the primary invariants, we find the exact quantum motions for the position and the momentum operators satisfying the Heisenberg equation of motion. The primary invariants give a very simple method to find the quantum motions. We also studied the quantum properties of the eigenstates and the coherent states for the LR-type invariant. As in the case of the single parametric oscillator [11–13], the classical solutions give all the descriptions of the corresponding quantum system.

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